DOCUMENT RESUME

ED 098 054 SE 018 075

AUTHOR Kidder, F. Richard
TITLE An Investigation of Nine-, Eleven-, and Thirteen-Year-Old Children's Comprehension of Euclidean Transformations.
PUB DATE 74
NOTE 25p.; Paper presented at the Annual Meeting of the National Council of Teachers of Mathematics (Atlantic City, New Jersey, 1974)

EDRS PRICE MF-$0.75 HC-$1.85 PLUS POSTAGE
DESCRIPTORS *Cognitive Development; *Elementary School Mathematics; *Geometric Concepts; Geometry; Mathematics Education; *Research; Secondary School Mathematics; Student Characteristics
IDENTIFIERS *Transformation Geometry

ABSTRACT

This study investigated middle school children's ability to form a mental image of a planar figure and to: (1) mentally flip, turn, or slide this representation and then construct the resultant image in correct position ("performing individual motions"), (2) perform two such motions in succession ("composition of motions"), (3) perform the process in reverse ("inverse motions"), and (4) hold length invariant while attempting the above three operations. The sample consisted of 30 students at the upper three-fourths ability level in each of grades 4, 6, and 8. Five tests were given to each student: (1) a 20-item paper-and-pencil pretest to determine ability to perform transformations at the perceptual-recognition level; (2) an individual test to see if students could physically perform flips, turns, and slides; and (3) a three-part transformation test, measuring performance on individual motions, compositions, and inverses. Five Age X Motion ANOVAs were used to analyze the five tests. Age was significant only in the pretest and the individual motions test. Slides were easier to perform than either flips or turns. A majority of subjects in each group failed to perform individual motions, compositions, and inverse motions; the majority of the errors on each task came from the failure to conserve length. (DT)

An Investigation of Nine-, Eleven-, and Thirteen-Year-Old Children's Comprehension of Euclidean Transformations

F. Richard Kidder
Longwood College
Farmville, Virginia

A Paper Presented at the Annual 1974 NCTM Meeting,
Atlantic City, New Jersey

Substantive recommendations for the inclusion of transformational geometry in the elementary and middle school curriculum have been made and are being implemented. However, there is a paucity of research data on which to base such inovations. More specifically, little empirical evidence exists concerning the spatial capability of children with respect to Euclidean Transformations. Therefore, the primary purpose of the present study was to investigate middle school children's comprehension of Euclidean transformations at the representational level; that is, to investigate their ability to form a mental image of a planar figure, to perform a mental operation (Euclidean transformation) on this representation, and then to construct the resultant in correct position. As suggested by Shulman (1970), in formulating the study, primary consideration was given to the structure of geometry and to a comparison of this structure with what is known about the cognitive structure of the middle school child. Klein's principle, with its emphasis on transformations, invariants under transformations, and groups of transformations concisely reveals the structure of geometry and is descriptive, in part at least, of cognitive operational structures as defined by the developmental theory of Jean Piaget.

Mathematical Background

In 1872, Felix Klein synthesized the vast geometrical knowledge of his day with Arthur Caley's algebraic invariants and Sophus Lie's transformational groups into the now famous Erlangen Program, which not only defines geometries, but also classifies them as to content. A modern restatement of Klein's principle is: "A geometry is the study of the properties of a set S which remain invariant when elements of S are subjected to the transformations of some transformational group." One is led to a

classification of geometries illustrated by the nesting of the defining transformational groups in the Venn diagrams of Figure 1. An understanding

Insert Figure 1 about here

of the role of the transformational groups is enhanced by an examination of the transformations and some of the basic properties which remain invariant under such transformations. Figure 2 lists the most important properties being held invariant by the transformational groups. These eleven properties are not intended to be exhaustive but do, in a very real sense, exemplify the major differences in the geometries.

Insert Figure 2 about here

Psychological Background

Three major themes are evident in Piaget's research on the child's development of space. First, Piaget's major focus in space development is space representation, not space perception. Second, he believes that these spatial representations are built up through the organization of mental "actions" performed on objects in space. And third, he claims that the child's earliest spatial concepts are topological in nature and that his projective and Euclidean concepts are concomitant extensions of these topological concepts. The first two, representational space and the organization of mental actions, are particularly relevant to the present study.

Laurendeau and Pinard (1970) give credit to Piaget for introducing the notion that a fundamental distinction exists between the various levels of knowledge implicit in the individual's contacts with the spatial world surrounding him. There are, according to Piaget, distinctions between

perceptual, sensorimotor and representational space. Representational space is space in which the child can imagine or mentally reconstruct an action, and where the schema of these actions has sufficient mobility and flexibility for the internalized actions to become fully reversible mental operations. Piaget and Inhelder (1967) explain the distinction thus:

> "The evolution of spatial relations proceeds at two different levels. It is a process which takes place at the perceptual level and at the level of thought or imagination. ...Perception is the knowledge of objects resulting from direct contact with them. As against this, representation or imagination involves the evocation of objects in their absence or, when it runs parallel to perception, in their presence" (pp. 3, 17).

To illustrate, very young babies can distinguish between circular shapes and triangular shapes, but it is not until much later that they can represent these figures to themselves in thought.

Firmly rooted in Piagetian theory are the words "actions" and "operations." Piagetians argue that cognition at all ages is best characterized as the application of real actions by the child. In infancy, these actions are externalized and observable for the most part, e.g., bringing thumb to mouth. As the child grows, the actions become more and more internalized, schematic, mobile, and less dependent upon concrete qualities. Gradually the cognitive actions coalesce to form systems of actions with strong structures. When they achieve this status they are called "cognitive operations" by Piaget (Flavel, 1963).

During middle childhood (7-11 years of age) there is an integration of the child's cognitive operations into system totalities with certain

definite properties--reversibility, composition, and associativity--which suggested to Piaget corresponding logico-mathematical group and lattice structures. Hence, he posited that a hybrid of the mathematical group and lattice structure was a good model for the structure formed by the concrete-operational child's system of cognitive operations.

As the child grows into adolescence, his concrete operational thought structures are extended by synthesizing the two forms of reversibility, inversion and reciporcity, into a single system of transformations (Inhelder and Piaget, 1958). Flavell (1963) cites that now the logico-mathematical structures which serve as abstract models for the child's thought processes, "...consist of integrated lattice-group structures, not just partial and incomplete lattice and group properties, as in the concrete operational groupings, but a full and complete lattice and a full and complete group, both integrated within one total system (pp.211-212)." Piaget claims, "That the adolescent's behavior in certain problem solving situations attests to a cognitive structure with the properties of a four-group...," consisting of the operation of I, the identity; N, the negation; R, the reciprocal; and C, the correlative (i.e., INRC group) (Flavell, 1963, p. 215). Inhelder and Piaget (1958) summarize their discussion of the adolescent's thought processes while solving mechanical equilibrium problems by noting that the INRC group comes into play in both its logical and physical form; i.e., the logical INRC form governs the propositional operations which the subject uses to describe and explain reality while the physical INRC form is revealed in the manner by which thought processes must be structured in order to solve problems.

In summary, representational space in the child develops very slowly. In the beginning the child is capable of only crude internalized actions. Gradually, he is able to perform mental operations not only on objects which are real and physically present but also on objects whose presence is only imagined. The child's mental representation becomes not just merely a recall from his memory bank, but active reconstructions of objects at the symbolic level.

The Study

The proceeding theoretical discussion suggests that the projective and Euclidean spatial framework of middle school children has developed sufficiently to enable them to perform Euclidean transformations at the representational level. Therefore, Piagetian-like tasks were developed to investigate the child's ability to: (1) form mental representations of planar figures, perform mental operations (Euclidean transformations) on these representations, and then construct the image of the transformations in proper position (hereafter referred to as performing individual motions); (2) perform two such processes in succession (compositions of motions); (3) perform the process in reverse (inverse motions); and (4) hold length invariant while attempting the above three operations. Three experimental hypotheses were formulated to investigate the above: (1) the capability to perform Euclidean transformations, compositions of transformations, and inverse transformations is age related; (2) the thirteen year-old adolescent's cognitive structure is such that he can perform Euclidean transformations, compositions of transformations, and inverse transformations; and (3) perceptual spatial ability, as measured by a paper and pencil test, is r lated to the ability to perform Euclidean transformations, compositions

of transformations and inverse transformations.

Sample

The sample consisted of ninety subjects--15 each from grades four, six, and eight in the Commerce, Georgia and Winder, Georgia School Systems. Mean ages by grades as of November 1, 1972 were 9.3, 11.3, and 13.3 respectively. All age levels were heterogeneous with regard to sex and race. In selecting the age group, consideration was given to Piaget's theory of developmental stages. Students identified by teachers as being in the lower one-fourth general ability level were excluded. As a post-hoc procedure, the Otis-Lennon Mental Ability Test Form J was adminstered to all subjects. The mean raw scor s for grades four, six, and eight were 43.6, 52.9, and 46.1 respectively.

Pretest

To determine subjects' ability to perform transformations at the perceptual-recognition level, a 20-item paper and pencil pretest was administered in groups by grade. There were four slide items and eight each of the flip and turn items. The pretest was patterned after Alonzo's (1970) Spatial Analogies Problems Test in which she assumed "...that two cognitive capabilities are needed to solve the test items: (1) the ability to identify the spatial transformation, and (2) the ability to reconstruct and select an identical transformation from the alternatives (p. 17.)." A prototypical pre-test item is given in Figure 3.

Insert Figure 3 about here

Operational Definition Activities

Each subject was individually given an "operational" definition of

translations (slides), reflections (flips), and rotations (turns). On the average, these activities required 10-15 minutes per subject. Using rigid figures and wire models of the motion indicators--slide-arrow, flip-line, and turn-arrow--the three motions were operationally defined thus: (a) an original figure and a motion indicator were placed in front of the subject, (b) a congruent copy was placed on top of the original figure, and (c) the motion was performed using the copy, leaving the original figure fixed. Flipping (turning, sliding) the copy, leaving the original fixed, lessened the reliance on memory of the position of the original, enhanced the probable use of symmetry and other properties inherent in Euclidean transformations, and assisted the testor in pointing out errors in the subject's performance. After a motion was demonstrated several times with various orientations, the subject was asked to perform the transformation as indicated by the motion indicator. A subject was considered operational if he performed three consecutive slides (flips, turns) correctly. If he made a mistake, his errors were discussed and he started over. After a total of five incorrect slides (flips, turns), he was considered non-operational on that particular motion. If a subject was found to be non-operational on all three motions, he was not tested further and was considered as having failed the entire transformational test. The rationale for such termination was based on the premise that if a child could not physically perform a motion with a rigid congruent copy of the original figure, then he would not be able to construct an image of the original figure under a motion.

Transformational Test

Immediately following the "operational definition" activities, those subjects found to be operational (72 of the 90 participants) were given the transformational test. The transformational test examined subjects' performance on individual motions, compositions of motions, and inverse motions at the representational level. The test was composed of tasks patterned after the protocols used by Piaget and his collaborators in their investigation of the child's development of space (Piaget and Inhelder, 1967). The three basic Euclidean transformations are slides (S), flips (F), and turns (T). Combinatorily these can be composed two at a time in nine ways (1) S-F, (1a) F-S, (2) S-T, (2a) T-S, (3) F-T, (3a) T-F, (4) S-S, (5) F-F, and (6) T-T. Using these compositions, the ordering of the tasks in the transformational test was as follows. For each subject, composition tasks 1 or 1a, 2 or 2a, and 3 or 3a were randomly selected and randomly ordered. Composition tasks, 4, 5, and 6 were randomly ordered next, followed by the random ordering of the inverse tasks on slides, flips, and turns.

In a composition task, a subject was given an original figure and two indicated motions as exemplified in Figure 4. Ten 1/8" diameter sticks,

Insert Figure 4 about here

six for construction of two congruent images of the original figure and four of differing lengths as distractors, were placed adjacent to the compositional task. To insure that each subject was aware that he could construct an intermediate image, leave the intermediate image fixed, then construct the final image, and that he could measure (compare sticks with

sides of original figure) he was told, "There are enough sticks to make two copies of the original triangle and you can measure if you want to." He was then asked to use the sticks and construct a copy of the original figure as it would look to him after performing the composition. The subject's construction of the final image was judged to be in one of three categories: (1) subject's composition was correct; (2) subject failed to conserve length, making a non-congruent image (with a non-congruent image, the kind and amount of errors were normally impossible to determine), and (3) subject conserved length making a congruent image, but he failed to perform the composition (he either constructed the image in incorrect position or changed the orientation of angles).

Regardless of success/failure on each composition, to examine capability with individual motions, each subject was asked to separately and individually construct the images of the two motions comprising the composition. After making the image of the first motion, the subject's construction was removed and a rigid congruent copy of the original figure was placed in correct intermediate position. The subject was then asked to construct the image of the second motion using the intermediate copy as the original figure of the second motion. To test the subject's capability with inverse motions, he was given a motion indicator and the image under the motion (but not the original figure). He was told, "This is how the figure looks to you after you have performed the indicated slide (flip, turn). Make the figure as it looked to you before it was slid (flipped, turned)."

Data Analysis and Findings

An Age X Motion repeated measurements analysis of variance was used on the pretest, operational definition activities, and compositional tasks

(consisting of three subtests: individual motions, compositions, and inverse motions) to determine if Age and/or Motion were significant factors in the subject's ability to perform Euclidean transformations (Greenhouse and Geisser, 1959). The ANOVA (Table 1) reveals that Motion was a significant factor ($p < .01$) in all modes of testing. Surprisingly, only in the pretest

- - - - - - - - - - - - - - - - - -

Insert Table 1 about here

- - - - - - - - - - - - - - - - - -

and the individual motions test was Age found to be significant ($p < .05$). Age X Motion Interaction was not found to be significant in any subtest.

Duncan's New Multiple Range Test was used as a post hoc procedure to isolate the above differences. Slides were found to be easier to perform than either flips or turns, but no significant difference was indicated between flips and turns. No significant pairwise contrasts were found in age. Clearly (see Table 2), the results did not indicate that the lack of age significance was due to the three age groups already having spatial capability to perform transformations at the representational level: the low scores on the transformational tasks indicated just the opposite. The only logical explanation is that lack of age significance was due to the fact that few children of the three age groups were able to satisfactorily perform transformations at the presentational level.

A descriptive analysis was used to assess subjects' ability to perform transformations. It was found that subjects had difficulty in performing transformations in all modes of testing. In particular, as shown in Table 2, the majority of subjects in each age group failed to perform individual motions, compositions of motions, and inverse motions at the representational level.

Insert Table 2 about here

Pearson's product moment correlation coefficients were computed to examine relationships between the spatial ability levels of performing transformations. The pretest scores (perceptual ability) were not highly related with performance on the transformational tasks (representational ability). However, the data summarized in Table 3 indicates that a marked relatedness did exist between the scores on the Otis-Lennon Mental Ability

Insert Table 3 about here

Test and the ability to perform transformations at the representational level. A comparison of the thirteen-year-old lower quartile subjects' performance with that of the nine-year-old upper quartile subjects' performance on the transformational tasks is striking. The thirteen-year-old lower quartile subjects were only able to satisfactorily perform one individual motion out of a possible 84 and no compositions or inverse motions. In contrast, the nine-year-old upper quartile subjects performed 20 individual motions, 7 compositions, and 3 inverse motions. The contrast in relatedness of pretest scores with performance on the transformational tasks and the Otis-Lennon Test scores with performance on the transformational tasks indicates that the pretest measured specific spatial abilities whereas the Otis-Lennon Test measured more general reasoning abilities. Furthermore, it is suggestive that, in addition to spatial ability, broad general reasoning ability is needed to perform Euclidean transformations at the representational level. These findings are, in general, confirmatory of Alonzo's analysis of the relation of spatial to "logical reasoning" abilities (1970).

In addition to the analyses performed in examining questions related to experimental hypotheses 1, 2, and 3, an error analysis was undertaken to determine the kinds of errors committed during the transformational tasks and to identify possible error trends by individuals. No error pattern could be established, either on motions or by individuals. However, it was found that 70% of individual motion errors, 67% of composition errors, and 59% of inverse motion errors were "failure to conserve length" (did not construct a congruent image of the original figure). When the percentages of length errors were averaged across motions, it was found that 69%, 71%, and 70% of the 9-, 11-, and 13-year-old subjects respectively, committed length errors on the individual motion tasks. Strikingly, only four of the 72 subjects attempting the transformational tasks were able to perform all 12 individual motions with no conservation of length errors.

Discussion

The data did not support the experimental hypothesis that adolescents could perform Euclidean transformations, compositions of transformations, and inverse transformations at the representational level. The findings are not consistent with the theoretical base for the study nor with results of previous studies. Piaget (1967) contends that by the age of nine or thereabout the child has the framework appropriate for comprehensive Euclidean and projective systems. Replications of Piaget's spatial tasks by Dodwell (1969) and Lovell (1959) note some variation in age but, in general support Piaget's position. Laurendeau and Pinard's (1970) findings are also supportive of Piagetian theory; however, they observed the emergence of the coordination of perspective ability to occur slightly later.

Little research on the child's ability to perform Euclidean transformations as such is available. St. Clair (1968) found that fourth and sixth grade students can learn the concepts of symmetry (reflections and rotations). In a training study using students from grades four and five, Turner (1967) reached the broad general conclusion that the teaching of symmetry and related principles has potential for increasing children's spatial capabilities. And, Williford (1970) found that second and third grade children could be taught transformational concepts but there was little transfer to other spatial abilities.

The above comparison forms a basis for the experimentor's conjecture that the ability to perform transformations at the representational level derives from formal-operational thought (in a Piagetian sense) and that the thirteen-year old subjects of the present study were not in the formal operational stage. This conjecture is supported in part by the preponderance of "failure to conserve length" errors committed by all age groups during the transformational tasks.

Failure to conserve length (construct a congruent image of the original figures) was the most prevalent cause for non-performance of the transformational tasks. Piaget's experiemental technique in examing the child's conservation of length (classical definition) consisted of showing the child two sticks of the same length laying side by side, asking him if they were equal in length, sliding one of the sticks forward approximately 1 cm, and again asking the child to say which of the two sticks was longer or if they were still the same length (Piaget, Inhelder, and Szeminska, 1960, p. 95). Piaget does not give the ages of the children he found in each stage; however, the protocols indicate that six to eight years of age was the range for

emergence of consistent conservation of length. Numerous studies also imply that young children, ages 6-8, do in fact conserve length. Shantz and Smock (1966), Divers (1970), and Carey and Steffe's (1972) findings infer that young children conserve length in the classical sense.

The evidence is strongly supportive of the proposition that adolescents do conserve length. Why, then, was this capability not evidenced during the present study? Piaget and others established a child's capability to conserve length with two sticks as described above. The studies referenced inferred length conservation from the acquisition of some other conservation capability such as transitivity or seriation. In contrast, the present study used three types of transformations, six different compositions, and three types of inverse transformations. And, instead of simple short sticks, length had to be conserved with a complex point-set planar figure. That is, the subject had to hold invariant the shape and size of different prescribed three-sided figures under much more complicated transformations than just a 1 cm slide of one stick. In the "classical" test, a simple comparison could be made between the stick which was moved and the stick which was not moved. No such comparison of corresponding sides of the original and image figures could be made in the present study. A subject had to be aware of the necessity to hold length invariant at the outset, and he had to select the proper length sticks with which to construct the image figure prior to making the transformations.

In addition to the large number of "failure to conserve length" errors, the almost complete lack of error pattern also suggests that the ability to perform transformations at the representational level derives from formal-operational thought. One general property of formal thought is the ability

to separate factors not given by direct observation and the necessary tools for holding one factor constant in order to determine causal action of another (Flavell, 1963, pp. 204-211). It appears that just such a capability is needed to perform Euclidean transformations at the representational level. The many factors involved, the noting of factors not given by direct observation, and the holding constant of some factors while varying others is best seen by example. To rotate a triangle as depicted in Figure 5 and to satisfactorily construct the turn-image, a subject must (at least): (1) be

---

Insert Figure 5 about here

---

aware that the length of the sides AB, AC, and BC remain invariant under the turn; (2) be able to imagine the existance of rays OD and OE, and the existance of angle ACD; (3) be aware that angle ACD is held invariant under the turn; (4) be able to rotate the figure through angle DOE; and (5) hold distances OC, OB, and OA invariant while rotating the triangle. In Figure 5, rays OD and OE are dashed, indicating that these are not given--that they must be imagined by the subject. If the subjects were at the concrete-operational state of reasoning (or even in a transitional stage) it is to be expected that they would focus attention, first on one variable and then on another in performing the transformational tasks. This would, of course, explain why length was sometimes conserved and sometimes not, and why errors on previous tasks of the same motion were not necessarily repeated on a current task.

The inability of subjects in the present study to perform Euclidean transformations at the representational level, particularly the failure to conserve length, should not be interpreted as a refutation of Piaget's theory

of the child's spatial development. The subjects were not tested on conservation of length in the "classical" sense, nor were they tested to ascertain if they were in the formal-operational state. The variances between the findings of the present study and Piagetian theory and between the findings of the present study and previous studies serve to highlight the need for further research in the area of transformational geometry.

BIBLIOGRAPHY

Alonzo, Martha R. "Spatial Analogies Problems Test, A Preliminary Validity Study." Unpublished M.A. Thesis, University of Georgia, 1970.

Carey, Russell L. and Steffe, Leslie P. "An Investigation in the Learning of Relational Properties by Kindergarten Children." Paper presented at the Annual 1972 AERA Meeting, Chicago, Illinois.

Divers, Benjamin P., Jr. "The Ability of Kindergarten and First Grade Children to Use the Transitive Property of Three Length Relations in Three Perceptual Situations." Unpublished Ed.D. Dissertation, University of Georgia, 1970.

Dodwell, Peter C. "Children's Perception and Their Understanding of Geometric Ideas." Studies in Cognitive Development. Edited by David Elkind and John H. Flavell, New York: Oxford University Press, 1969.

Flavell, John H. The Developmental Psychology of Jean Piaget. New York: Van Nostrand Reinhold Company, 1963.

Greenhouse, S. W. and Ceisser, Seymour. "On Methods in the Analysis of Profile Data." Psychometrica. XXIV (June, 1959), 95-112.

Inhelder, Barbel and Piaget, Jean. The Growth of Logical Thinking from Childhood to Adolescence. New York: Basic Books, Inc., 1958.

Laurendeau, Monique and Pinard, Adrian. The Development of the Concept of Space in the Child. New York: International University Press, Inc., 1970.

Lovell, Kenneth. "A Follow Up Study of Some Aspects of the Work of Piaget and Inhelder in The Child's Conception of Space." British Journal of Educational Psychology, XXIX (June, 1959), 104-112.

Otis, Arthus S. and Lennon, Roger T. Otis-Lennon Mental Ability Test. New York: Harcourt, Brace and World, Inc., 1967.

Piaget, Jean and Inhelder, Barbel. The Child's Conception of Space. New York: W. W. Norton and Company, Inc., 1967.

Piaget, Jean; Inhelder, Barbel; and Szeminska, A. The Child's Conception of Geometry. New York: Harper and Row, 1960.

Shantz, Carolyn U. and Smock, Charles D. "Development of Distance Conservation and the Spatial Coordinate System." Child Development. XXXVII (December, 1966), 943-948.

Shulman, Lee S. "Psychology and Mathematics Education." Mathematics Education. The Sixty-ninth Yearbook of the National Society for the Study of Education. Edward G. Begle, editor, Chicago: University of Chicago Press, 1970.

St. Clair, Irene Zercher. "A Study of the Development of the Concept of Symmetry by Elementary Children." Unpublished Ph.D. Dissertation, University of Texas at Austin, 1968.

Turner, Mervyn Leonard. "The Learning of Symmetry Principles and Their Transfer to Tests of Spatial Ability." Unpublished Ed.D. Dissertation, University of California, Berkeley, 1967.

Williford, Harold J. "A Study of Transformational Instruction in the Primary Grades." Unpublished Ed.D. Dissertation, University of Georgia, 1970.

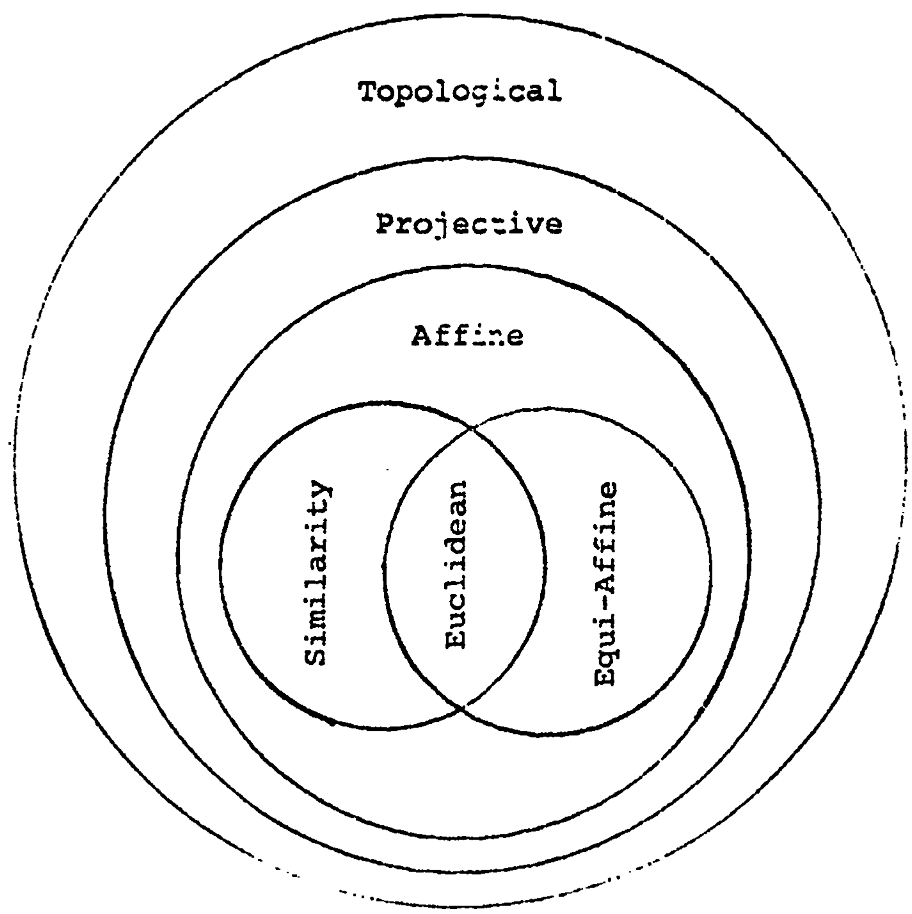
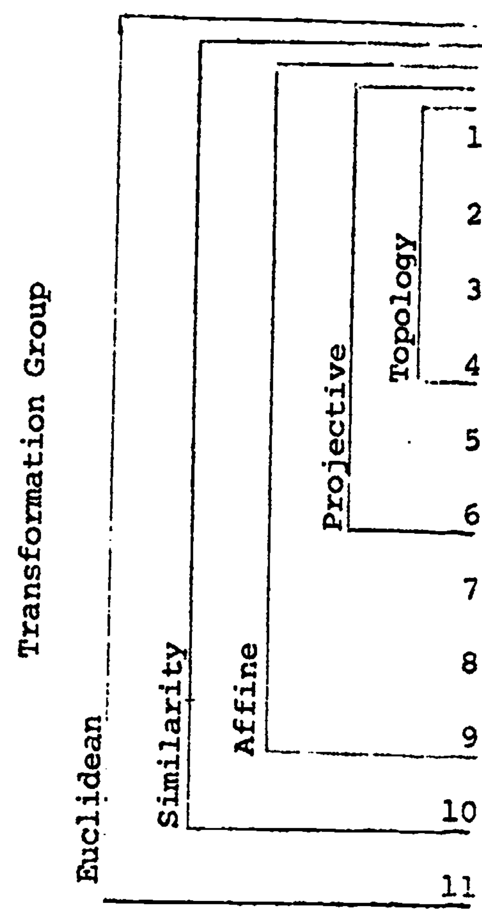


Fig. 1. Venn Diagram: Nesting of Transformational Groups

Invariant Properties



1. openness (closedness) of curves
2. interior, exterior, boundary point
3. linear order, cyclic order
4. connectedness
5. collinearity and concurrency
6. cross-ratio
7. parallelism of lines
8. ratio of distance
9. betweenness of points
10. measure of angles
11. length

Fig. 2. Properties Remaining Invariant Under Transformations

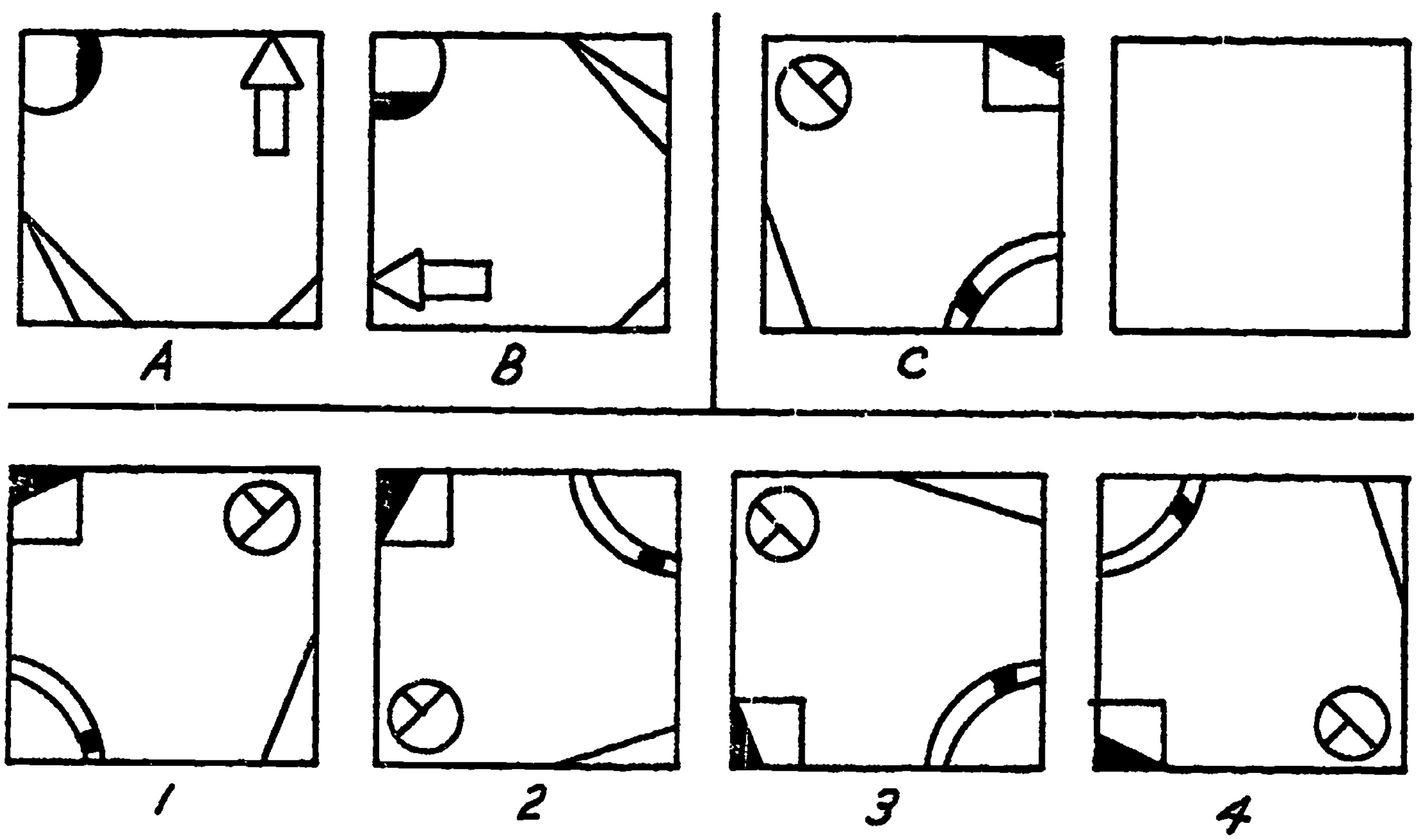


Fig. 3. Prototypical Pretest Item

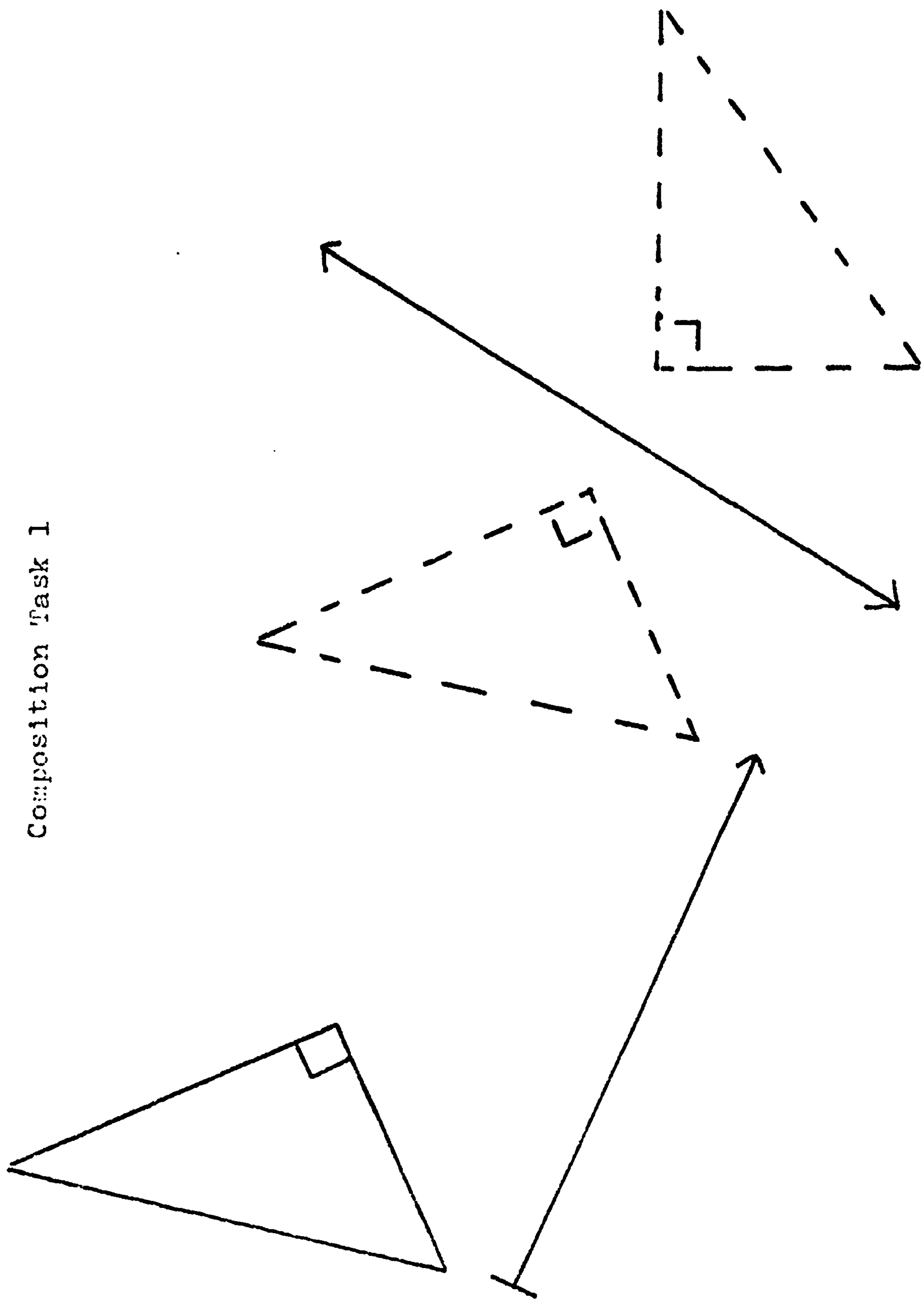


Fig. 4. Typical Composition Task

Table 1

Summary of Analysis of Variance: Five Tests Comprising the Experimental Tasks

	SV	DF	F	
Pretest	Age	2	7.2968	$p < .01$
	Motion	2	11.1753	$p < .005$
	A X M	4	0.5855	
Operational Definition	Age	2	3.6569	
	Motion	2	28.3902	$p < .001$
	A X M	4	1.2666	
Individual Motion	Age	2	4.1723	$p < .05$
	Motion	2	22.3591	$p < .001$
	A X M	4	1.1634	
Composition	Age	2	0.9667	
	Composition	5	8.5368	$p < .01$
	A X C	10	0.2477	
Inverse Motion	Age	2	3.8737	
	Motion	2	10.8685	$p < .005$
	A X M	4	2.2432	

Table 2

Performance on Transformational Test by Age Group

Age	Slides		Flips		Turns	
	Number and Percent of Motions Performed*					
9	24	20.0%	11	9.2%	8	6.7%
11	34	28.3%	20	16.7%	19	15.8%
13	53	44.2%	30	25.0%	23	19.2%
Total	111	30.8%	61	16.9%	50	13.9%

*There were 120 slides, flips, and/or turns possible for each age group.

Table 3

Upper Quartile (Q_1) Versus Lower Quartile (Q_4)

Performance on Experimental Tasks [a]

Age[b]		Otis-Lennon Mean Scores	No. individual motions performed[c]	No. composition performed[d]	No. Inverse motions performed[e]
9	Q_1	68	20	7	3
	Q_4	25.1	5	1	1
11	Q_1	68.3	35	7	9
	Q_4	36.3	5	2	2
13	Q_1	63.9	37	8	11
	Q_4	28.6	1	0	0
Total	Q_1		92	22	23
	Q_4		11	3	3

a. as determined by score on Otis-Lennon Mental Ability Test.

b. seven subjects in each quartile per age.

c. twelve individual motions possible for each subject.

d. six compositions possible for each subject.

e. three inverse motions possible for each subject.